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Fully developed laminar natural convection in open-ended vertical concentric annuli

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Abstract—Analytical solutions for fully developed natural convection in open-ended vertical concentric annuli are presented. Four fundamental boundary conditions have been investigated and the corresponding fundamental solutions are obtained. These four fundamental boundary conditions are obtained by combining each of the two conditions of having one boundary maintained at uniform heat flux or at uniform wall temperature with each of the conditions that the opposite boundary is kept isothermal at the inlet fluid temperature or adiabatic. Expressions for flow and heat transfer parameters are given for each case. These fundamental solutions may be used to obtain solutions satisfying more general thermal boundary conditions.

INTRODUCTION

LAMINAR free convection in vertical open-ended channels is likely to find wider use as it could provide the flow mechanism in some types of solar heating and ventilating passive systems. In modern electronic equipment, the vertical circuit boards include heat generating elements and this situation can be modelled by parallel heated plates with upward flow in the intervening space. Examples of other applications which may be simulated by such a model are the external surface of electric transformers, small domestic mobile winter oil heaters and some types of radiators of hydronic heating systems.

Since the work of Elenbaas [1], free convective flow through vertical plane channels and pipes has received quite extensive attention. Reviews of previous work done on these two geometries may be found in the paper by Quintiere and Mueller [2] and the recent publications by Pollard and Oosthuizen [3], Oosthuizen [4], and Wirtz and Haag [5].

On the other hand, the annular geometry is widely employed in the field of heat exchangers. A typical application is that of gas cooled nuclear reactors, in which cylindrical fissionable fuel elements are placed axially in vertical coolant channels within the graphite moderator, the cooling gas flowing along the channels parallel to the fuel elements. Generally, heat transfer within such channels is by turbulent forced convection, but laminar free convection may provide the sole means of the necessary cooling during shut down periods.

Developing laminar natural convection in vertical concentric annuli has been studied by El-Shaarawi and Sarhan [6, 7], Al-Arabi *et al.* [8], Oosthuizen and Paul [9] and El-Shaarawi and Khamis [10]. In two of these investigations [8, 10] a constant heat flux is applied at one of the boundaries of the annulus while the opposite boundary is insulated. In other investigations [6, 7, 9] the case of an isothermal boundary and an opposite adiabatic boundary has been considered. Moreover, Oosthuizen and Paul [9] have also considered the case of two isothermal boundaries one of which is maintained at the ambient temperature. All these investigations use boundary-layer assumptions, which are applicable at large Rayleigh numbers. The obtained results show that at relatively low Rayleigh numbers, or sufficiently large height to gap width ratios (l/b), fully developed conditions can be achieved before the fluid reaches the annulus exit cross-section.

Fully developed free convection flows are obtained when the inertia forces vanish and a balance is attained between the pressure and gravitational forces on the one hand and the viscous forces on the other hand. The study of such flows gives the limiting conditions for developing flows and provides an analytical check on numerical solutions.

Nevertheless, to the authors' knowledge, only two studies [7, 11] are available in the literature dealing with fully developed free convection flows in vertical annuli. For annuli with one isothermal boundary and an opposite adiabatic boundary, El-Shaarawi and Sarhan [7] showed that the fully developed free convection axial velocity profile is similar to that for axial forced flows in annuli. In a recent publication, Joshi [11] presented an analytical solution for the fully developed free convection flow in annuli with two isothermal boundaries, the inner of which is maintained at a higher temperature than the outer one.

The lack of analytical solutions for fully developed laminar natural convection in vertical concentric annuli with constant heat flux boundary conditions

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NOMENCLATURE

a local heat transfer coefficient based on the area of the heat transfer surface, $q/(t_w - t_0) = \mp k(\delta t/\delta r)_w/(t_w - t_0)$, minus and plus signs apply respectively for heating and cooling at the inner boundary and vice versa at the outer boundary

$\bar{a}$ average heat transfer coefficient over the annulus height, based on the average temperature of the heat transfer boundary, $\bar{h}/\pi D_w l(\bar{t}_w - t_0) = \int_0^l a\,dz/l$

b annular gap width, $(r_2 - r_1)$

C_i constants of integrations where $i = 1, 2, 3$ and 4

c_p specific heat of fluid at constant pressure

D equivalent (hydraulic) diameter of annulus, $2b$

D_w diameter of heat transfer boundary

f volumetric flow rate, $\int_{r_1}^{r_2} 2\pi r u\,dr = \pi(r_2^2 - r_1^2)u_0$

F dimensionless volumetric flow rate, $f/\pi l\gamma\, Gr^*$

g gravitational body force per unit mass (acceleration)

Gr Grashof number, $\mp g\beta(t_w - t_0)D^3/\gamma^2$ in the case of an isothermal boundary or $\mp g\beta q D^4/2\gamma^2 k$ in the case of uniform heat flux (UHF) heat transfer boundary, the plus and minus signs apply to upward (heating) and downward (cooling) flows, respectively. Thus Gr is a positive number in both cases

Gr^* modified Grashof number, $D\,Gr/l$

h heat gained or lost by fluid from the entrance up to a particular elevation in the annulus, $\rho_0 f c_p(t_m - t_0)$ for both cases ($\pi D_w q z$ in the case of uniform heat flux at the heat transfer boundary)

$\bar{h}$ heat gained or lost by fluid from the entrance up to the annulus exit, i.e. value of h at $z = 1$, $\rho f c_p(\bar{t}_m - t_0)$ and in the case of UHF at the heat transfer boundary this reduces to $\pi D_w l q$

H dimensionless heat absorbed from the entrance up to any particular elevation, $h/[\pi\rho_0 c_p l\gamma\, Gr^*\,(t_w - t_0)]$ in the case of an isothermal heat transfer boundary or $2hk/\pi\rho_0 c_p\gamma\, Gr^*\, qDl$ in the case of the UHF heat transfer boundary and reduces in both cases to FT_m, i.e. $2\int_N^1 URT\,dR$

$\bar{H}$ dimensionless heat absorbed from the entrance up to the annulus exit, i.e. value of H at $z = 1$, $\bar{h}/[\pi\rho_0 c_p l\gamma\, Gr^* \times (t_w - t_0)]$ in the case of an isothermal heat transfer boundary or $2\bar{h}k/(\pi\rho_0 c_p\gamma\, Gr^* \times qDl)$ in the case of the UHF heat transfer boundary and reduces in both cases to $F\bar{T}_m$, i.e. $2\int_N^1 UR\bar{T}\,dR$

k thermal conductivity of fluid

l height of annulus

L dimensionless height of annulus, l/Gr^*

N annulus radius ratio, r_1/r_2

Nu local Nusselt number, $|a|D/k$

$\bar{Nu}$ average Nusselt number based on the area of the heat transfer surface over the whole annulus height, $|\bar{a}|D/k$, it reduces to $2/\bar{T}_w$ in the case of UHF at the heat transfer boundary

p pressure of fluid inside the channel at any cross-section

p' pressure defect at any point, $p - p_s$

p_0 pressure of fluid at the annulus entrance

p_s hydrostatic pressure, $\mp \rho_0 g z$ where the minus and plus signs are for upward (heating) and downward (cooling) flows, respectively

P dimensionless pressure defect at any point, $p' r_2^4/\rho_0 l^2\gamma^2\, Gr^{*2}$

P_0 dimensionless pressure defect at annulus entrance, $p_0 r_2^4/\rho_0 l^2\gamma^2\, Gr^{*2}$

Pr Prandtl number, $\mu c_p/k$

q heat flux at the heat transfer surface; it has positive values in the case of upward (heating) flows and negative values in the case of downward (cooling) flows, $q = \mp k(\partial t/\partial r)_w$ where the minus and plus signs are, respectively, for heating and cooling in case I. These signs should be reversed in Case O

Q_l heat transfer per unit length

r radial coordinate

r_1 inner radius of annulus

r_2 outer radius of annulus

R dimensionless radial coordinate, r/r_2

Ra Rayleigh number, $Gr\,Pr$

Ra^* modified Rayleigh number, $Gr^*\,Pr$

t fluid temperature at any point

t_m mixing cup temperature over any cross-section, $\int_{r_1}^{r_2} rut\,dr/\int_{r_1}^{r_2} ru\,dr$

$\bar{t}_m$ mixing cup temperature at exit cross-section, i.e. value of t_m at $z = 1$

t_0 fluid temperature at annulus entrance

t_w temperature of heat transfer boundary

$\bar{t}_w$ average temperature of heat transfer boundary

T dimensionless temperature at any point, $(t - t_0)/(t_w - t_0)$ in the case of an isothermal heat transfer boundary or $(t - t_0)/(qD/2k)$ for UHF at the heat transfer boundary and thus it is positive for both heating (upward) and cooling (downward) flows

NOMENCLATURE (*continued*)

- T_m dimensionless mixing cup temperature at any cross-section, $(t_m - t_0)/(t_w - t_0)$ in the case of an isothermal heat transfer boundary and $(t_m - t_0)/(qD/2k)$ in the case of UHF at the heat transfer boundary
- $\bar{T}_m$ mixing cup temperature at exit cross-section, i.e. value of T_m at $z = l$
- T_w temperature of heat transfer boundary at any cross-section
- $\bar{T}_w$ temperature of heat transfer boundary at exit cross-section, i.e. value of T_w at $z = l$
- u axial velocity component at any point
- u_0 entrance axial velocity, $\int_{r_1}^{r_2} 2\pi r u \, dr / [\pi(r_2^2 - r_1^2)]$
- U dimensionless axial velocity component at any point, $u r_2^2 / l \gamma Gr^*$
- z axial coordinate
- Z dimensionless axial coordinate, $z / l\, Gr^*$.

Greek symbols

- β volumetric coefficient of thermal expansion
- γ kinematic viscosity of fluid, μ/ρ_0
- μ dynamic viscosity of fluid
- ρ fluid density at temperature t, $\rho_0[1 - \beta(t - t_0)]$
- ρ_0 fluid density at inlet fluid temperature t_0.

and also with isothermal boundary conditions other than those treated in refs. [7, 11], motivated the present work. The purpose of this paper is to present, in closed forms, fully developed free convection solutions, corresponding to four fundamental thermal boundary conditions, in vertical concentric annuli.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

We consider steady laminar fully developed free convection flow inside an open-ended vertical concentric annulus of a finite length (l), immersed in a stagnant fluid of infinite extent maintained at a constant temperature t_0. Figure 1 shows the physical situation in which at least one of the channel walls is heated or cooled either isothermally or at a constant wall heat flux so that its temperature (i.e. temperature of the inner surface of the outer cylinder or that of the outer surface of the inner cylinder) is different from the ambient temperature t_0. Due to fully developed flow assumptions the fluid enters the part under consideration of the annular passage with an axial velocity profile which remains invariant in the entire channel (i.e. $\partial u/\partial z = 0$). The fluid is assumed to be Newtonian, enters the channel at the ambient temperature t_0, and has constant physical properties but obeys the Boussinesq approximation according to which its density is constant except in the gravitational term of the vertical momentum equation. Axial symmetry is assumed and viscous dissipation and internal heat generation are absent.

Under the above mentioned assumptions and using the dimensionless parameters given in the Nomenclature, the equations of continuity, motion and energy reduce to the following two simultaneous non-dimensional equations:

$$-\frac{dP(Z)}{dZ} + \frac{1}{R}\frac{d}{dR}\left[R\frac{dU(R)}{dR}\right] + \frac{T(R,Z)}{16(1-N)^4} = 0 \quad (1)$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial T(R,Z)}{\partial R}\right] = Pr\, U(R)\frac{\partial T(R,Z)}{\partial Z}. \quad (2)$$

Four boundary conditions are therefore needed to obtain a solution for the above two second-order differential equations. The two conditions related to U are

$$U(1) = U(N) = 0. \quad (3)$$

On the other hand, there are many possible thermal boundary conditions applicable to the annular configuration. In the present paper, the non-dimensional parameters used in the formulation of the problem are chosen to suit annuli having their two boundaries at two different uniform heat fluxes (q_1 and q_2) or at two different uniform temperatures (t_1 and t_2) or annuli under one of four fundamental boundary conditions. These four fundamental boundary conditions are obtained by combining each of the two conditions of having one boundary maintained at uniform wall heat flux (q) or at uniform wall temperature (t_w) with each of the conditions that the opposite boundary is kept isothermal at the inlet fluid temperature (t_0) or adiabatic ($\partial t/\partial r = 0$).

With the two boundaries of an annulus maintained at UHF conditions, if q_1 refers to the higher heat flux then q_1 will be at the hotter wall in case of heating and at the cooler wall in case of cooling. Thus, the value of r_q (ratio of the heat fluxes at the two boundaries, q_2/q_1) may vary from -1 to 1. Similarly, when the two boundaries of an annulus are kept isothermal, t_1 refers to the wall which has the larger temperature difference from t_0. Thus, t_1 is the temperature of the hotter wall in case of heating the two boundaries and of the cooler wall in case of cooling both boundaries. Therefore, the wall temperature difference ratio $r_T[= (t_2 - t_0)/(t_1 - t_0)]$ may, in this case of UWT boundary conditions, also vary from -1 to 1.

From the previous discussion it may be seen that there are many thermal boundary conditions appli-

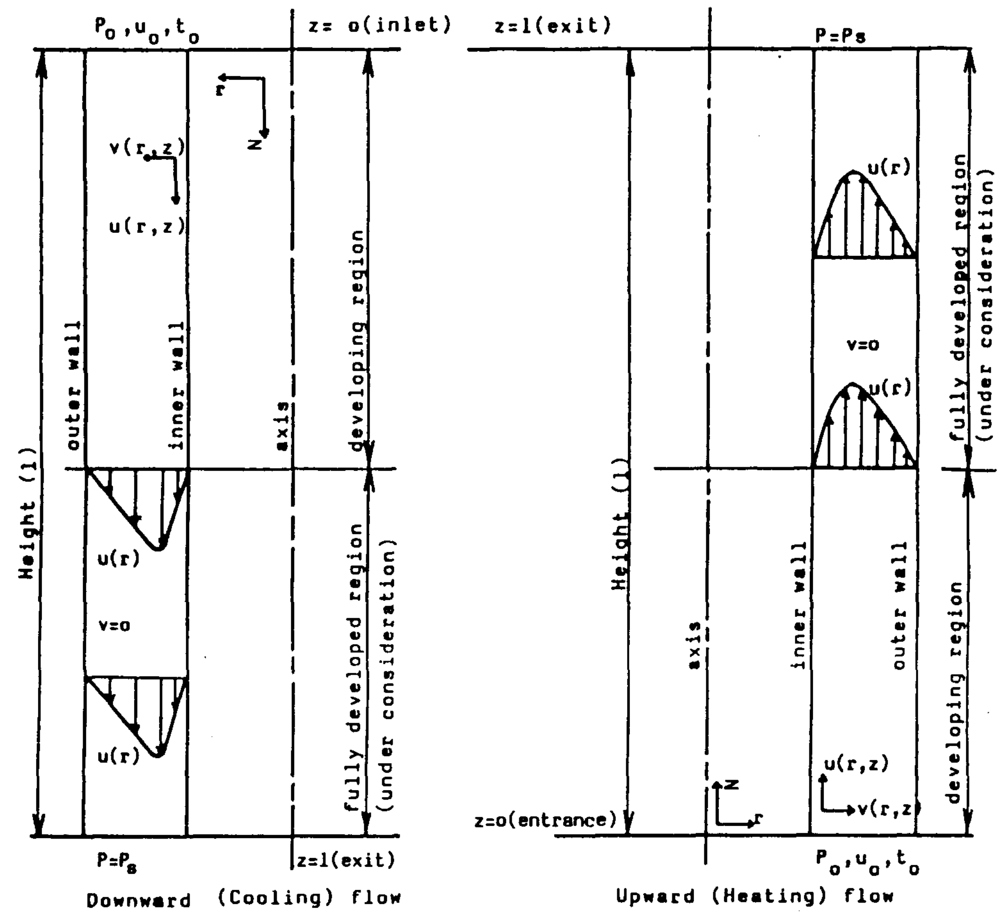

FIG. 1.

cable to the annular geometry. However, under certain conditions, the energy equation (2) becomes linear and homogeneous in T (e.g. when $\partial T/\partial z$ is constant), and then any linear combination of solutions will be a solution. It may then be possible to develop certain special or fundamental solutions to this equation satisfying particular or specific boundary conditions, which can be combined to satisfy any other boundary conditions. This method is known as the method of superposition. Reynolds *et al.* [12] defined four fundamental boundary conditions for the annular geometry which produce four fundamental solutions to the energy equation (2) when it becomes linear. For the sake of completeness, these fundamental solutions are stated hereinafter.

(1) Fundamental solutions of first kind, which satisfy the boundary conditions of a temperature step change at one wall, the opposite wall being kept isothermal at the inlet fluid temperature. Using the present notation, this corresponds to $T = 1$ at one wall and $T = 0$ at the opposite wall (i.e. $r_T = 0$).

(2) Fundamental solutions of the second kind which satisfy the boundary conditions of a step change in heat flux at one wall, the opposite wall being adiabatic. Using the present notation, this corresponds to $\partial T/\partial R = -1/(1-N)$ at the inner wall and $\partial T/\partial R = 0$ at the outer wall or $\partial T/\partial R = 0$ at the inner wall and $\partial T/\partial R = 1/(1-N)$ at the outer wall ($r_q = 0$).

(3) Fundamental solutions of the third kind which satisfy the boundary conditions of a temperature step change, at one wall, the opposite wall being adiabatic. This corresponds to $T = 1$ at one wall and $\partial T/\partial R = 0$ at the opposite wall.

(4) Fundamental solutions of the fourth kind which satisfy the boundary conditions of a step change in heat flux at one wall, the opposite wall being kept isothermal at the inlet fluid temperature. This corresponds to $\partial T/\partial R = -1/(1-N)$ at the inner wall while $T = 0$ at the outer wall or $T = 0$ at the inner wall and $\partial T/\partial R = 1/(1-N)$ at the outer wall.

With any of the above mentioned boundary conditions, the boundary opposite to that maintained adiabatic (i.e. $\partial T/\partial R = 0$) or at t_0 (i.e. $T = 0$) is termed the heat transfer boundary (even though there is transfer of heat through a boundary maintained at $T = 0$). For each of the above fundamental solutions, two cases are considered, namely, case (I), in which the heat transfer boundary is at the inner wall and case

(O) in which the heat transfer boundary is at the outer wall. The aim of the present paper is to obtain the above mentioned four fundamental solutions.

GENERAL ANALYSIS

Substituting T from equation (1) into equation (2), we obtain

$$U\,Pr\frac{\mathrm{d}^2P}{\mathrm{d}Z^2}+\frac{\mathrm{d}^4U}{\mathrm{d}R^4}+\frac{2}{R}\frac{\mathrm{d}^3U}{\mathrm{d}R^3}-\frac{1}{R^2}\frac{\mathrm{d}^2U}{\mathrm{d}R^2}+\frac{1}{R^3}\frac{\mathrm{d}U}{\mathrm{d}R}=0. \tag{4}$$

A solution of equation (4) in the form $U = U(R)$ is only possible if

$$\mathrm{d}^2P/\mathrm{d}Z^2=\alpha \tag{5}$$

where α is a constant. From equation (1) we then have

$$\partial T/\partial Z = 16\alpha(1-N)^4 \tag{6}$$

which means that, for a given R in a given annulus, the dimensionless temperature T varies linearly with the axial distance Z. This implies that the assumption of a hydrodynamically fully developed free convection flow should necessarily mean that the flow is also thermally fully developed, regardless of the value of the Prandtl number (Pr). In other words, for free convection flows in vertical channels, the thermal development length is shorter than or at most equal to that of the hydrodynamic development length, irrespective of the value of the Prandtl number. However, in pure forced convection flows, such a result is only obtained if $Pr \leqslant 1$.

Integrating equation (5) twice and applying the conditions that $P = 0$ at both inlet and exit (i.e. at $Z = 0$ and L), gives

$$P=\alpha Z(Z-L)/2. \tag{7}$$

Substituting equation (5) into equation (4) yields

$$R^4\frac{\mathrm{d}^4U}{\mathrm{d}R^4}+2R^3\frac{\mathrm{d}^3U}{\mathrm{d}R^3}-R^2\frac{\mathrm{d}^2U}{\mathrm{d}R^2}+R\frac{\mathrm{d}U}{\mathrm{d}R}+\lambda^4R^4U=0 \tag{8}$$

where

$$\lambda^4=\alpha\,Pr. \tag{9}$$

Equation (8) may also be written in the following concise form:

$$\left[\frac{\mathrm{d}}{\mathrm{d}R^2}+\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\right]^2U+\lambda^4U=0. \tag{10}$$

Substituting P from equation (7) into equation (1) gives

$$\alpha\left(Z-\frac{L}{2}\right)-\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}U}{\mathrm{d}R}\right)=\frac{T}{16(1-N)^4}. \tag{11}$$

The two governing equations (10) and (11) can be simplified if one of the two annulus boundaries is kept isothermal. In order to satisfy this boundary condition, the left-hand side of equation (11) must, in this particular case, be independent of Z. Thus, it is concluded that α (and hence λ) must, in such a case, equal zero. Therefore, equations (5)–(8) and (11) reduce, in this case, to the following equations, respectively:

$$\alpha=\frac{\mathrm{d}^2P}{\mathrm{d}Z^2}=0 \tag{12}$$

$$\frac{\partial T}{\partial Z}=0 \tag{13}$$

$$P=0 \tag{14}$$

$$R^3U''''+2R^2U'''-RU''+U'=0 \tag{15}$$

$$T=-\frac{16(1-N)^4}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}U}{\mathrm{d}R}\right). \tag{16}$$

Equation (13) states that, in a case with an isothermal boundary, the fully developed temperature profile is constant or at most a function of the radial coordinate only. On the other hand, equation (14) states that the fully developed pressure inside an annulus with an isothermal boundary is equal to the hydrostatic pressure, at the same elevation, outside the annulus. This implies that, in such a fully developed case with an isothermal boundary, there would be no pressure drop due to fluid viscous drag since this latter is just offset by the bouyancy driving force.

If the two governing equations (15) and (16) or their general forms (8) and (11) are solved for the velocity and temperature profiles (U and T) then the following useful parameters can be evaluated.

The dimensionless volumetric flow rate (F) can be evaluated from the following equation:

$$F=2\int_N^1 RU\,\mathrm{d}R. \tag{17}$$

Since for a fully developed flow U is a function of R only, it follows that the definite integral on the right-hand side of equation (17) and hence F are constants regardless of the value of the axial coordinate Z, i.e. they are also constants irrespective of the value of the channel height. It can, however, be shown that, in cases with two UHF boundaries, there exists a relation between this constant fully developed value of F and the thermal boundary conditions applied at the boundaries of an annulus. Integrating equation (2) with respect to R from $R = N$ to 1, the following equation is obtained:

$$\left(R\frac{\partial T}{\partial R}\right)_{R=1}-\left(R\frac{\partial T}{\partial R}\right)_{R=N}=Pr\int_N^1 RU\frac{\partial T}{\partial Z}\mathrm{d}R. \tag{18}$$

However, for a given annulus, equation (6) shows that, in cases with two UHF boundaries, $\partial T/\partial Z$ is constant and hence it can be taken out of the above integral. Substituting for $\partial T/\partial Z$ from equation (6) in equation (18) and using the result in equation (17) gives

$$F = 2\int_N^1 RU\,\mathrm{d}R$$
$$= \left[\left(\frac{\partial T}{\partial R}\right)_{R=1} - N\left(\frac{\partial T}{\partial R}\right)_{R=N}\right]\Big/[8\lambda^4(1-N^4)]. \quad (19)$$

It may also be worth mentioning that, in a case with a UWT heat transfer boundary, equations (13) and (18) give the following result:

$$\left(\frac{\partial T}{\partial R}\right)_{R=1} = N\left(\frac{\partial T}{\partial R}\right)_{R=N}. \quad (20)$$

Taking into consideration that the rate of heat transfer per unit length from the inner and outer surfaces of an annulus are given, respectively, by

$$Q_{li} = \mp 2\pi r_1 k\left(\frac{\partial t}{\partial r}\right)_{r=r_1} \quad (21)$$

and

$$Q_{lo} = \pm 2\pi r_2 k\left(\frac{\partial t}{\partial r}\right)_{r=r_2} \quad (22)$$

then for a fundamental solution of the first kind (i.e. a case with two UWT boundaries), we have

$$Q_{li} = \mp 2\pi k(t_w - t_0)N\left(\frac{\partial T}{\partial R}\right)_{R=N} \quad (23a)$$

and

$$Q_{lo} = \pm 2\pi k(t_w - t_0)\left(\frac{\partial T}{\partial R}\right)_{R=1}. \quad (23b)$$

The upper and lower (plus or minus) signs in the above expressions apply, respectively, for heating and cooling. Equations (20), (23a) and (23b) yield the following conclusions. In an annulus with two UWT boundaries, or an annulus with a UWT boundary and an opposite UHF boundary, when fully developed conditions are achieved, the rate of heat transfer from one boundary should be equal and opposite to that from the other boundary (i.e. $A_1q_1 = -A_2q_2$). This implies that, in such cases, the net rate of heat transfer to/from the fully developed fluid flow is zero. Thus, it is anticipated, in such cases, that the bulk fluid temperature would remain constant. This is because the heat passes through the fluid from one boundary to the other in such a manner as if the fluid were stationary, i.e. by pure steady conduction. However, in the special case with a UWT boundary ($T = 1$) and an opposite adiabatic boundary ($\partial T/\partial R = 0$), equation (20) shows that ($\partial T/\partial R$) at the UWT boundary must also vanish. Thus, in this special case, fully developed conditions are achieved when both ($\partial T/\partial R$) and ($\partial T/\partial Z$) vanish, i.e. the temperature becomes uniform at the UWT boundary.

Equation (19) confirms that the fully developed dimensionless volumetric flow rate is independent of the dimensionless channel height (L) and it depends on the thermal boundary conditions applied at the two annulus boundaries. This means that, when the channel becomes sufficiently high so that the flow reaches its state of full development, a further increase in the channel height would not produce any further increase in the sucked volumetric flow rate. When fully developed conditions are achieved, in a case with two UHF boundaries, an increase in the value of F may be obtained by increasing the heat flux at the boundaries rather than the channel height L.

The dimensionless inlet velocity U_0 is given in terms of F by

$$U_0 = F/(1-N^2). \quad (24a)$$

Therefore, U_0 is similarly constant irrespective of the annulus height and is related, in cases with UHF boundaries, to the thermal boundary conditions by the following equation:

$$U_0 = \left[\left(\frac{\partial T}{\partial R}\right)_{R=1} - N\left(\frac{\partial T}{\partial R}\right)_{R=N}\right]\Big/[8\lambda^4(1-N)^5(1+N)]. \quad (24b)$$

The dimensionless mixing cup temperature is given by

$$T_m = \int_N^1 UTR\,\mathrm{d}R\Big/\int_N^1 UR\,\mathrm{d}R. \quad (25)$$

To find the variation of T_m, in the fully developed flow region, with the dimensionless axial distance Z, the above equation is differentiated with respect to Z. Since U is independent of Z, this gives

$$\frac{\mathrm{d}T_m}{\mathrm{d}Z} = \int_N^1 \frac{\partial T}{\partial Z}UR\,\mathrm{d}R\Big/\int_N^1 UR\,\mathrm{d}R$$

which, on substituting for $\partial T/\partial Z$ from equation (4) into the above equation, yields

$$\frac{\mathrm{d}T_m}{\mathrm{d}Z} = 16\alpha(1-N)^4. \quad (26)$$

Integrating equation (26) with respect to Z between channel entrance and exit, taking into consideration that $T_m = 0$ at $Z = 0$, results in

$$T_m = 16\alpha(1-N)^4 Z. \quad (27)$$

Using the dimensionless parameters given in the Nomenclature, the following expressions for the local Nusselt number can easily be obtained:

with a UWT boundary

$$Nu = \pm 2(1-N)(\partial T/\partial R)_w; \quad (28a)$$

with two UHF boundaries

$$Nu = \pm 2(1-N)(\partial T/\partial R)_w/T_w = 2/T_w \quad (28b)$$

where the minus and plus signs apply respectively for cases (I) and (O) when there is heating and vice versa when there is cooling.

From equation (16) it can be seen that ($\partial T/\partial R$) is a function of R only which is dependent on the fully developed axial velocity profile (U), i.e. it is independent of Z. Hence, for a case with a UWT heat transfer boundary, equations (16) and (28a) show that the fully developed local Nusselt number is constant, i.e. independent of Z. Consequently, the fully developed average Nusselt number is, in this case (UWT), constant, i.e. independent of channel height L. On the other hand, with the two boundaries at UHF, equation (11) shows that the temperature varies linearly with Z. Hence, equations (11) and (28b) show that the fully developed local Nusselt number and consequently the average Nusselt number, for a given annulus with UHF boundaries, vary hyperbolically with Z. These conclusions are as expected since, as was previously mentioned, the assumption of hydrodynamically fully developed flows implies also thermally fully developed free convection flows.

FUNDAMENTAL SOLUTIONS OF FIRST KIND

In this case, the two boundaries of the annulus are kept isothermal, one of which is at the inlet ambient fluid temperature t_0 while the opposite boundary is at a higher or a lower temperature. Therefore, equations (15) and (16) are the two governing equations in such a case.

Equation (15) is readily solved and we find

$$U = C_1 \ln R + C_2 R^2/2 + C_3(R^2 \ln R/2 - R^2/4) + C_4 \tag{29}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Substituting from equation (29) into equation (16) we obtain

$$T = A + B \ln R \tag{30}$$

where

$$A = -16(1-N)^4(2C_2+C_3), \quad B = -32(1-N)^4 C_3.$$

To obtain the constants C_1, C_2, C_3 and C_4, the velocity boundary conditions (3) together with the following thermal boundary conditions should be applied to equations (29) and (30).

Case (I). Temperature step at the inner wall while the outer wall is kept at the ambient temperature, i.e.

$$T(N,Z) = 1 \quad \text{and} \quad T(1,Z) = 0.$$

Case (O). Temperature step at the outer wall while the inner wall is kept at the ambient temperature, i.e.

$$T(N,Z) = 0 \quad \text{and} \quad T(1,Z) = 1.$$

The solutions obtained are as follows.

Case (I)

$$U = [(N^2 \ln N + 1 - N^2) \ln R/\ln N + R^2 \ln R - 1 - R^2]/[64(1-N)^4 \ln N] \tag{31a}$$

and

$$T = \ln R/\ln N. \tag{32a}$$

Case (O)

$$U = [(1+\ln N)(1-R^2) + R^2 \ln R + (N^2 - \ln N - 1)\ln R/\ln N]/[64(1-N)^4 \ln N] \tag{31b}$$

and

$$T = 1 - \ln R/\ln N. \tag{32b}$$

It may be worth mentioning that each of the temperature profiles (32a) and (32b) satisfies equation (20), which provides a check on these obtained solutions.

The volume flow rate F and the non-dimensional mixing cup temperature T_m, defined by equations (17) and (25) are readily calculated using solutions obtained for U and T. The results are as follows.

Case (I)

$$F = \left[-\frac{3}{16} - \frac{N^2}{4} + \frac{7N^4}{16} - \frac{N^4 \ln N}{4} + \frac{N^2}{2\ln N} - \frac{1}{4\ln N} - \frac{N^4}{4\ln N}\right] \Big/ [32(1-N)^4 \ln N] \tag{33a}$$

$$T_m = \frac{2}{F}\left[\frac{5}{32} - \frac{3N^2}{4} + \frac{27N}{32} + \frac{N^2 \ln N}{4} - \frac{3N^4 \ln N}{8} + \frac{1}{4\ln N} - \frac{N^4 (\ln N)^2}{4}\right] \Big/ [64(1-N)^4 (\ln N)^2]. \tag{34a}$$

Case (O)

$$F = \left[\frac{11}{16} - \frac{N^2}{4} - \frac{3N^4}{16} + \frac{\ln N}{4} - \frac{N^2}{2\ln N} + \frac{N^4}{4\ln N} + \frac{1}{4\ln N}\right] \Big/ [32(1-N)^4 \ln N] \tag{33b}$$

$$T_m = \frac{2}{F}\left[\frac{10}{16} - \frac{N^2}{4} - \frac{N^2}{2\ln N} + \frac{21}{32\ln N} + \frac{\ln N}{4} - \frac{N^2}{2(\ln N)^2} + \frac{1}{4(\ln N)^2} - \frac{13N^4}{32\ln N} + \frac{N^4}{4(\ln N)^2}\right] \Big/ [64(1-N)^4 \ln N]. \tag{34b}$$

It may be worth mentioning that, in the present case (with isothermal boundaries), the temperature T (and hence T_m) does not vary with axial distance Z. Thus $\bar{T}_m = T_m$ and $\bar{H} = H$. This means that the heat transferred to/from the fluid through the two boundaries of the annulus, under the fully developed flow conditions, does not affect the fluid bulk temperature since they are equal and opposite (in order that fully developed conditions can be achieved in such a case).

Expressions for the fully developed Nusselt number (local and also average) are obtained after getting the temperature gradient at the walls from equations

(32a) and (32b) and then substituting in equation (28a). The results are:

for case (I)

$$Nu_I = -2(1-N)/(N\ln N); \tag{35a}$$

for case (O)

$$Nu_O = -2(1-N)/\ln N. \tag{35b}$$

Finally, an expression for the average Nusselt number over the entire annulus height (i.e. from entrance until exit, taking the developing region into consideration) in terms of Rayleigh number, when fully developed conditions are achieved, can be derived by substituting values of $\bar{H}$ in the following equation:

$$\overline{Nu} = D\bar{H}\,Ra^*/D_w \tag{35c}$$

i.e.

for case (I)

$$\overline{Nu}_I = (1-N)\bar{H}\,Ra^*/N; \tag{35d}$$

for case (O)

$$\overline{Nu}_O = (1-N)\bar{H}\,Ra^*. \tag{35e}$$

FUNDAMENTAL SOLUTIONS OF SECOND KIND

In this case, one of the annulus boundaries is maintained at a constant heat flux (q) and the opposite boundary is perfectly insulated. The governing equations in such a case are equations (8) and (11). Equation (8) has the following solution in terms of Bessel functions of zero order:

$$U = C_1^1 J_0(\sqrt{i}\lambda R) + C_2^1 Y_0(\sqrt{i}\lambda R) + C_3^1 I_0(\sqrt{i}\lambda R) + C_4^1 K_0(\sqrt{i}\lambda R).$$

This solution can be expressed in terms of Kelvin functions of zero order as

$$U = C_1\,\mathrm{ber}(\lambda R) + C_2\,\mathrm{bei}(\lambda R) + C_3\,\mathrm{ker}(\lambda R) + C_4\,\mathrm{kei}(\lambda R) \tag{36}$$

where

$$\mathrm{ber}(\lambda R) \mp i\,\mathrm{bei}(\lambda R) = J_0(\lambda R i^{\mp 3/2}) = I_0(\lambda R i^{\pm 1/2})$$

$$\mathrm{ker}(\lambda R) \pm i\,\mathrm{kei}(\lambda R) = K_0(\lambda R i^{\pm 1/2})$$

$$K_0(\lambda R) = \tfrac{1}{2}\pi i[I_0(\lambda R) + i Y_0(i\lambda R)]$$

J_0 and Y_0 are Bessel functions of zero order while I_0 and K_0 are the modified Bessel functions of zero order. Tables of Kelvin functions ber, bei, ker, kei and also Bessel functions J_0, Y_0, I_0 and K_0 are available in ref. [13].

Taking into consideration that

$$[R\,\mathrm{ber}'R]' = -R\,\mathrm{bei}\,R, \quad [R\,\mathrm{bei}'R]' = R\,\mathrm{ber}\,R$$

$$[R\,\mathrm{ker}'R]' = -R\,\mathrm{kei}\,R, \quad [R\,\mathrm{kei}'R]' = R\,\mathrm{ker}\,R$$

where $'$ means differentiation with respect to R, substitution of U from equation (36) into equation (13) yields the following solution for the temperature profile:

$$\frac{T}{16(1-N)^4} = \alpha\left(Z - \frac{L}{2}\right) + \lambda^2[C_1\,\mathrm{bei}(\lambda R) - C_2\,\mathrm{ber}(\lambda R) + C_3\,\mathrm{kei}(\lambda R) - C_4\,\mathrm{ker}(\lambda R)]. \tag{37}$$

The constants C_1, C_2, C_3 and C_4 are evaluated as follows. Taking into consideration the following relations:

$$\sqrt{2}\,\mathrm{ber}'R = \mathrm{ber}_1 R + \mathrm{bei}_1 R$$

$$\sqrt{2}\,\mathrm{bei}'R = -\mathrm{ber}_1 R + \mathrm{bei}_1 R$$

$$\sqrt{2}\,\mathrm{ker}'R = \mathrm{ker}_1 R + \mathrm{kei}_1 R$$

$$\sqrt{2}\,\mathrm{kei}'R = -\mathrm{ker}_1 R + \mathrm{kei}_1 R$$

and differentiating equation (37) with respect to R we get

$$\frac{\sqrt{2}}{16\lambda^3(1-N)^4}\frac{\partial T}{\partial R} = C_1[-\mathrm{ber}_1(\lambda R) + \mathrm{bei}_1(\lambda R)] - C_2[\mathrm{ber}_1(\lambda R) + \mathrm{bei}_1(\lambda R)] + C_3[-\mathrm{ker}_1(\lambda R) + \mathrm{kei}_1(\lambda R)] - C_4[\mathrm{ker}_1(\lambda R) + \mathrm{kei}_1(\lambda R)]. \tag{38}$$

The present fundamental thermal boundary conditions are given below.

Case (I). Step change in heat flux at the inner wall while the outer wall is adiabatic, i.e.

$$\partial T/\partial R|_{R=N} = -1/(1-N) \quad \text{and} \quad \partial T/\partial R|_{R=1} = 0.$$

Case (O). Step change in heat flux at the outer wall while the inner wall is adiabatic, i.e.

$$\partial T/\partial R|_{R=N} = 0 \quad \text{and} \quad \partial T/\partial R|_{R=1} = 1/(1-N).$$

Substituting the velocity boundary conditions (3) and the above thermal boundary conditions, equations (36) and (38), yield the following four equations in C_1, C_2, C_3 and C_4.

Case (I)

$$[M] \times \begin{vmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -\sqrt{2}/[16\lambda^3(1-N)^5] \\ 0 \end{vmatrix}.$$

Case (O)

$$[M] \times \begin{vmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2}/[16\lambda^3(1-N)^5] \end{vmatrix}$$

where $[M]$ is the matrix of coefficients given by

$$[M] = \begin{vmatrix} [\mathrm{ber}(\lambda N)] & [\mathrm{bei}(\lambda N)] & [\mathrm{ker}(\lambda N)] & [\mathrm{kei}(\lambda N)] \\ [\mathrm{ber}(\lambda)] & [\mathrm{bei}(\lambda)] & [\mathrm{ker}(\lambda)] & [\mathrm{kei}(\lambda)] \\ \begin{bmatrix} -\mathrm{ber}_1(\lambda N) \\ +\mathrm{bei}_1(\lambda N) \end{bmatrix} & \begin{bmatrix} -\mathrm{ber}_1(\lambda N) \\ -\mathrm{bei}_1(\lambda N) \end{bmatrix} & \begin{bmatrix} -\mathrm{ker}_1(\lambda N) \\ +\mathrm{kei}_1(\lambda N) \end{bmatrix} & \begin{bmatrix} -\mathrm{ker}_1(\lambda N) \\ -\mathrm{kei}_1(\lambda N) \end{bmatrix} \\ \begin{bmatrix} -\mathrm{ber}_1(\lambda) \\ +\mathrm{bei}_1(\lambda) \end{bmatrix} & \begin{bmatrix} -\mathrm{ber}_1(\lambda) \\ -\mathrm{bei}_1(\lambda) \end{bmatrix} & \begin{bmatrix} -\mathrm{ker}_1(\lambda) \\ +\mathrm{kei}_1(\lambda) \end{bmatrix} & \begin{bmatrix} -\mathrm{ker}_1(\lambda) \\ -\mathrm{kei}_1(\lambda) \end{bmatrix} \end{vmatrix}.$$

Each of the above two sets of equations can be solved by some standard procedure, such as Cramer's rule, to obtain the C's in terms of λ. The value of λ can be determined by the following procedure.

By definition, the dimensionless mixing cup temperature, in the present case, is given by the equation

$$T_{\mathrm{m}} = \frac{2}{Pr}\frac{D_{\mathrm{w}}}{D}\frac{Z}{F}. \tag{39}$$

Equating the right-hand sides of equations (27) and (39), the following expression for λ is obtained:

$$\lambda^4 = \alpha\, Pr = \frac{1}{8(1-N)^4 F}\frac{D_{\mathrm{w}}}{D}. \tag{40}$$

This expression reduces in case (I) to

$$\lambda^4 = \frac{N}{8(1-N)^5 F} \tag{41}$$

and in case (O) to

$$\lambda^4 = \frac{1}{8(1-N)^5 F}. \tag{42}$$

It may be worth mentioning that the same expressions, equations (41) and (42), can be obtained by substituting the present boundary conditions in equation (19). Using standard integral relations for the Kelvin functions, it may be shown that

$$F = 2\int_N^1 UR\,\mathrm{d}R = \frac{-\sqrt{2}}{\lambda}[R\{C_1[\mathrm{ber}_1(\lambda R) - \mathrm{bei}_1(\lambda R)] + C_2[\mathrm{bei}_1(\lambda R) + \mathrm{ber}_1(\lambda R)] + C_3[\mathrm{bei}_1(\lambda R) + \mathrm{ber}_1(\lambda R)] + C_4[\mathrm{kei}_1(\lambda R) + \mathrm{ker}_1(\lambda R)]\}]_N^1. \tag{43}$$

To obain λ and the values of C_1, C_2, C_3 and C_4 a simple iterative procedure may be used. An assumed initial value of λ is used to obtain an initial set of values for C's. These values are then used in equation (43) to obtain a value for F, and hence a second iterate for λ may be obtained from equation (41) or (42). The procedure is repeated until convergence within a specified tolerance is obtained.

Having obtained the value of λ (and hence α), equation (27) (equation (39)) can be used to obtain T_{m}.

Finally, the following expressions for the fully developed local Nusselt number are obtained after substituting the values of T_{w} from equation (37) in equation (28b):

for case (I)

$$Nu = 1/[8(1-N)^4[\alpha(Z-L/2) + \lambda^2\{C_1\,\mathrm{bei}(\lambda N) - C_2\,\mathrm{ber}(\lambda N) + C_3\,\mathrm{kei}(\lambda N) - C_4\,\mathrm{ker}(\lambda N)\}]]; \tag{44}$$

for case (O)

$$Nu = 1/[8(1-N)^4[\alpha(Z-L/2) + \lambda^2\{C_1\,\mathrm{bei}(\lambda) - C_2\,\mathrm{ber}(\lambda) + C_3\,\mathrm{kei}(\lambda) - C_4\,\mathrm{ker}(\lambda)\}]]. \tag{45}$$

FUNDAMENTAL SOLUTIONS OF THIRD KIND

In this case, since one of the boundaries is isothermal, equations (14)–(16) are the governing equations. However, since the wall opposite to the heat transfer surface is perfectly insulated, the fluid temperature in the annular space becomes ultimately uniform at the same temperature as the heated surface ($T = 1$). Thus, in this case, an isothermal flow (at the temperature of the heat transfer boundary), in which a balance is attained between the buoyancy and viscous forces, is achieved. Since fully developed conditions in this case mean that $P = 0$ and $T = 1$, in both case (I) and case (O), equation (1) reduces to

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left[R\frac{\mathrm{d}U}{\mathrm{d}R}\right] = -\frac{1}{16(1-N)^4} \tag{46}$$

which upon integrating twice and applying the velocity boundary conditions (3) gives

$$U = [1-R^2-(1-N^2)(\ln R/\ln N)]/[64(1-N)^4]. \tag{47}$$

For full development, at $T = 1$, the mixing cup temperature is also uniform everywhere ($T_{\mathrm{m}} = 1$). Therefore, values of the dimensionless volumetric flow rate (F) and the dimensionless heat absorbed over the entire channel height ($\bar{H}$) are equal and given, for both cases (I) and (O), by

$$F = \bar{H} = \frac{1-N^2}{128(1-N)^4}\left[1+N^2+\frac{1-N^2}{\ln N}\right]. \tag{48}$$

Since full development conditions yield an isothermal flow, the fully developed local Nusselt number is zero. However, expressions for the average Nusselt number over the entire annulus height can be obtained by substituting from equation (48) in equations (35d) and (35e). This gives

for case (I)

$$\overline{Nu}_{\mathrm{I}} = \frac{(1+N)}{128N(1-N)^2}\left[1+N^2+\frac{1-N^2}{\ln N}\right]Ra^*; \quad (49)$$

for case (O)

$$\overline{Nu}_{\mathrm{O}} = (1-N)\bar{H}\,Ra^*$$
$$= \frac{(1+N)}{128(1-N)^2}\left[1+N^2+\frac{1-N^2}{\ln N}\right]Ra^*. \quad (50)$$

FUNDAMENTAL SOLUTIONS OF FOURTH KIND

In this case, since one of the boundaries is isothermal, equations (14)–(16) are the governing equations subject to the following boundary conditions.

Case (I). Step change in heat flux at the inner wall while the outer wall is isothermal at the inlet fluid temperature, i.e.

$$\partial T/\partial R|_{R=N} = -1/(1-N) \quad \text{and} \quad T(1) = 0. \quad (51a)$$

Case (O). Step change in heat flux at the outer wall while the inner wall is isothermal at the inlet fluid temperature, i.e.

$$T(N) = 0 \quad \text{and} \quad \partial T/\partial R|_{R=1} = 1/(1-N). \quad (51b)$$

General solutions for equations (15) and (16) have already been obtained. These solutions are given by equations (29) and (30) which are also applicable here. However, the constants of integrations should be obtained by applying the velocity boundary conditions (3) together with the thermal boundary condition (51a) or (51b). Taking into consideration that the differentiation of equation (30) with respect to R gives

$$\partial T/\partial R = -32C_3(1-N)^4/R \quad (52)$$

the application of these boundary conditions yields the constants of integrations in the following sequence. Firstly, the value of C_3 is obtained from equation (52) after applying the boundary condition of a constant heat flux at the heat transfer boundary. Secondly, the value of C_2 is obtained from equation (30) after applying the condition of $T = 0$ at the opposite boundary. Thirdly, using equation (29) and the boundary condition $U(1) = 0$, the value of C_4 can be obtained. Finally, the boundary condition $U(N) = 0$ in equation (29) results in the value of C_1. The obtained values of C's are:

for case (I)

$$C_1 = N(N^2-N^2\ln N-1)/[64(1-N)^5\ln N],$$
$$C_2 = -N/[64(1-N)^5],$$
$$C_3 = -2C_2, \text{ and } \quad C_4 = -C_2;$$

for case (O)

$$C_1 = (1-N^2+\ln N)/[64(1-N)^5\ln N],$$
$$C_2 = (1+2\ln N)/[64(1-N)^5]$$
$$C_3 = -1/[32(1-N)^5], \text{ and } \quad C_4 = C_3/2.$$

The corresponding velocity and temperature profiles are:

for case (I)

$$U = \frac{N}{64(1-N)^5}\left[(N^2-N^2\ln N-1)\frac{\ln R}{\ln N} - R^2+R^2\ln R+1\right] \quad (53a)$$

$$T = -N\ln R/(1-N); \quad (54a)$$

for case (O)

$$U = \frac{1}{64(1-N)^5}\left[(1-N^2+\ln N)\frac{\ln R}{\ln N} + R^2(1+\ln N-\ln R)-\ln N-1\right] \quad (53b)$$

$$T = (\ln R-\ln N)/(1-N). \quad (54b)$$

Expressions for the fully developed dimensionless volumetric flow rate in both cases are:

for case (I)

$$F = \frac{N}{32(1-N)^5}\left[\frac{3}{16}+\frac{N^2}{4}+\frac{N^4}{4\ln N}-\frac{7N^4}{16} + \frac{N^4\ln N}{4}-\frac{N^2}{2\ln N}+\frac{1}{4\ln N}\right]; \quad (55a)$$

for case (O)

$$F = \frac{1}{64(1-N)^5}\left[-\frac{7}{16}+\frac{N^2}{4}+\frac{3N^4}{16}-\frac{1}{4\ln N} - \frac{\ln N}{4}+\frac{N^2}{2\ln N}-\frac{N^4}{4\ln N}\right]. \quad (55b)$$

The dimensionless heat absorbed from the entrance up to any particular cross-section at which the flow reaches its state of full development remains constant until the fluid reaches the exit cross-section (i.e. $H = \bar{H}$). This is because, in a fully developed region with an isothermal boundary (as in the present case), the heat gained through one boundary must be lost through the opposite boundary. Expressions for such a fully developed value of H (also $\bar{H}$) are:

for case (I)

$$\bar{H} = H = 2\int_N^1 UTR\,\mathrm{d}R = \frac{N^2}{32(1-N)^6}\left[\frac{5}{32}+\frac{N^2}{32} - \frac{21N^4}{32}+\frac{1}{4\ln N}-\frac{N^2}{2\ln N}+\frac{N^4}{4\ln N} + \frac{5N^4\ln N}{8}-\frac{N^4(\ln N)^2}{4}\right]; \quad (56a)$$

for case (O)

$$\bar{H} = H = 2\int_N^1 UTR\,\mathrm{d}R = \frac{1}{32(1-N)^6}\left[\frac{21}{32} - \frac{N^2}{2} - \frac{5N^4}{32} + \frac{10\ln N}{16} + \frac{1}{4\ln N} + \frac{(\ln N)^2}{4} - \frac{N^2}{4\ln N} + \frac{N^4}{4\ln N}\right]. \quad (56b)$$

Similarly, the mixing cup temperature remains constant from any cross-section at which the flow reaches a state of full development until the fluid reaches the exit cross-section (i.e. $\bar{T}_m = T_m$ at full development). Now, expressions for the fully developed mixing cup temperature T_m (also $\bar{T}_m$) can easily be obtained, for cases (I) and (O), by dividing equation (56a) over equation (55a) and equation (56b) over equation (55b), respectively.

Finally, substituting the temperature gradient at the wall, obtained from the differentiation of equation (54a) or (54b), in equation (28a) the value of the fully developed local Nusselt number in case (I) or case (O), respectively, is shown to be equal to 2.

CONCLUSIONS

Analytical solutions for fully developed upward (heating) or downward (cooling) natural convection velocity and temperature profiles in open-ended vertical concentric annuli have been obtained. These solutions correspond to four fundamental boundary conditions obtained by combining each of the two conditions of having one boundary maintained at UHF or at UWT with each of the conditions that the opposite boundary is kept adiabatic or isothermal at the inlet fluid temperature. Expressions for the fully developed volumetric flow rate, heat absorbed by fluid, fluid mixing cup temperature, and local Nusselt number are presented for each considered case. Such fully developed values are approached, in a given annulus, when the modified Rayleigh number (Ra^*) attains a considerably high value. These values represent the limiting conditions and provide analytical checks on numerical solution for developing flows. The results in refs. [6, 7] indicate that present analytical solutions are approached by the developing flows at large L.

Once a developing natural convection flow reaches a state of full development, in a given annulus, the volumetric flow rate reaches its upper value; any further increase in the annulus height would not produce an increase in the volumetric flow rate. Moreover, for cases with an isothermal boundary, in a given annulus, the Nusselt number reaches its lower limiting value while the heat absorbed by the fluid and the mixing cup temperature reach their upper limiting values and all remain constant irrespective of any further increase in the channel height. However, for cases with two UHF boundary conditions, in a given annulus, the wall temperature (T_w), the heat absorbed by the fluid and the mixing cup temperature continue their linear variations with further increases in the channel height.

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CONVECTION NATURELLE LAMINAIRE DANS UN ESPACE ANNULAIRE CONCENTRIQUE VERTICAL A BOUTS OUVERTS

Résumé—On présente des solutions analytiques pour la convection naturelle laminaire établie dans un espace annulaire concentrique vertical à extrémités ouvertes. On considère quatre conditions aux limites fondamentales et on obtient les solutions correspondantes. Ces quatre conditions sont obtenues en combinant celles de flux uniforme ou de température uniforme sur la paroi avec celles de la paroi opposée maintenue isotherme à la température d'entrée du fluide ou bien adiabatique. Des expressions sont données dans chaque cas pour les paramètres d'écoulement ou de transfert de chaleur. Ces solutions fondamentales peuvent être utilisées pour obtenir des solutions satisfaisant des conditions aux limites thermiques plus générales.

VOLL AUSGEBILDETE LAMINARE NATÜRLICHE KONVEKTION IN OFFENEN, SENKRECHTEN, KONZENTRISCHEN RINGRÄUMEN

Zusammenfassung—Die analytischen Lösungen für die voll ausgebildete natürliche Konvektion in offenen, senkrechten, konzentrischen Ringräumen werden dargestellt. Die Lösungen werden für grundlegende Randbedingungen ermittelt, welche durch Kombination von konstanter Temperatur oder konstantem Wärmestrom an der einen Wand mit adiabaten Bedingungen oder konstanter Temperatur (Eintrittstemperatur des Fluids) an der anderen Wand entstehen. Für alle vier Fälle werden Beziehungen für die Strömungs- und Wärmeübergangsparameter angegeben. Mit Hilfe dieser grundlegenden Lösungen können Lösungen für allgemeinere Randbedingungen abgeleitet werden.

ПОЛНОСТЬЮ РАЗВИТАЯ ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ ПРИ ЛАМИНАРНОМ ТЕЧЕНИИ В НЕЗАМКНУТЫХ ВЕРТИКАЛЬНЫХ КОНЦЕНТРИЧЕСКИХ КОЛЬЦЕВЫХ КАНАЛАХ

Аннотация—Приведены аналитические решения для полностью развитой естественной конвекции в незамкнутых вертикальных концентрических кольцевых каналах. Эти решения соответствуют четырем типам граничных условий. Указанные типы граничных условий представляют собой комбинации постоянного теплового потока или постоянной температуры на одной из стенок канала с постоянной температурой, равной температуре жидкости на входе, или отсутствием теплообмена на противоположной стенке. Получены выражения для параметров течения и теплопереноса, соответствующих каждому такому случаю. Приведенные к статье решения можно использовать для генерации решений, удовлетворяющих более общим тепловым граничным условиям.